# MEASURE THEORY ON GRAPHS

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Abstract. The main purpose of this paper is to introduce several measures determined by the given finite directed graph. We will introduce three measures $\mu_G$, $\mu_{G^{\wedge}}$ and $\mu$, for the given graph $G$. Those three measures are depending on the new algebraic structures, determined by $G$, so-called the diagram set, the reduced diagram set and the free semigroupoid, respectively. Let $\mathbb{F}^{+}(G)$ be the free semigroupoid of $G$ consisting of the empty word $\emptyset$, the vertex set $V(G)$ and the finite path set $FP(G)$, under the admissibility. For each element $w$ in $\mathbb{F}^{+}(G)$, we have the graphical image $\delta_w$ of $w$ on $\mathbb{R}^2$. The set $D(G)$ of such elements is called the diagram set and it is a quotient semigroupoid of $\mathbb{F}^{+}(G)$. On $D(G)$, we can define a $G$-measure $\mu_G = d \cup \Delta$, where $d$ is a weighted degree measure on $V(G)$ and $\Delta$ is the diagram length measure on $D(G) \setminus V(G)$. Then it is a positive bounded measure. For the given graph $G$, we can determine the suadowed graph $G^{\wedge} = G \cup G^{-1}$, where $G^{-1}$ is the opposite directed graph (or shadow) of $G$. Then similar to the previous construction, we can construct the free semigroupoid $\mathbb{F}^{+}(G^{\wedge})$ of $G^{\wedge}$ and the diagram set $D(G^{\wedge})$ of $G^{\wedge}$. Define the reducing relation on $D(G^{\wedge})$ and construct the reduced diagram set $D_r(G^{\wedge})$. Then it is a groupoid. On it, define the graph measure $\mu_{G^{\wedge}} = d \cup \Delta^r$, where $d$ is given as above and $\Delta^r$ is the reduced-diagram-length measure on $D_r(G^{\wedge}) \setminus V(G^{\wedge})$. Then it is also a bounded positive measure on $D_r(G^{\wedge})$. Futhermore, the measure $\mu_G$ can be regarded as a restricted measure of $\mu_{G^{\wedge}}$ on $\mathbb{F}^{+}(G) \subset \mathbb{F}^{+}(G^{\wedge})$. In Chapter 3, we observe the graph integral of graph measurable functions with respect to the graph measure $\mu_{G^{\wedge}}$. In Chapter 4, we introduce the restricted measure of the graph measure, so-called subgraph measures. In Chapter 5, we will extend the graph measure $\mu_{G^{\wedge}}$ as a measure $\mu$ defined on the free semigroupoid $\mathbb{F}^{+}(G^{\wedge})$ of $G^{\wedge}$. Different from the graph measure $\mu_{G^{\wedge}}$, this extended measure $\mu$ is a locally bounded positive measure. Also, we briefly consider the integration with respect to this measure.

Let $G$ be a finite directed graph, with its vertex set $V(G)$ and its edge set $E(G)$. Throughout this paper, we say that a graph is finite if $|V(G)| < \infty$ and $|E(G)| < \infty$. Let $v \in V(G)$ be a vertex. The vertex $v$ has degree $m$ if it has $m$-incident edges and denote it by $\deg(v) = m$. If $v$ is isolated without the incident edges, then define $\deg(v) = 0$. On the graph $G$, we can have a finite path $w = e_1\ e_2\ ...\ e_k$, where $e_1$, ..., $e_k$ are admissible directed edges in $E(G)$. Define the length of $w$ by $k$ and denote it by $|w| = k$. The length of a finite path is nothing but the cardinality of the admissible edges constructing the given finite path. Suppose the finite path $w$ has its initial vertex (or the source) $v_1$ and its terminal vertex (or the range) $v_2$. Then we write $w = v_1\ w\ v_2$ to emphasize the initial vertex and the terminal

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vertex of $w$. Sometimes, we will denote $w = v_1\ w$ (or $w = w\ v_2$), for emphasizing the initial vertex of $w$ (resp. the terminal vertex of $w$). Notice that the finite path $w$ can be expressed as the word in $E(G)$. Define the set $FP(G)$ of all finite paths. Then

$$FP(G) = \cup_{n=1}^{\infty} FP_n(G),$$

where

$$FP_n(G) \overset{def}{=} \{w : |w| = n\}.$$

Let $w_1$ and $w_2$ be in $V(G) \cup FP(G)$ and assume that the word $w_1\ w_2$ is again contained in $V(G) \cup FP(G)$. Then we say that $w_1$ and $w_2$ are admissible (or they are connected under the ordering (1, 2), where 1 and 2 are indices of $w_1$ and $w_2$). Note that, in general, even though $w_1$ and $w_2$ are admissible, $w_2$ and $w_1$ are not admissible. However, if $w_1$ and $w_2$ are loop finite paths and if $w_1$ and $w_2$ are admissible, then $w_2$ and $w_1$ are admissible, too. However, $w_1\ w_2 \neq w_2\ w_1$, in this case, because the directions of them are different and hence they are different finite path. We say that edges $e_1$, ..., $e_k$ are admissible if there exists a finite path $w = e_1\ ...\ e_k$ with $|w| = k$.

Define the free semigroupoid $\mathbb{F}^+(G) = \{\emptyset\} \cup V(G) \cup FP(G)$, with its binary operation $(\cdot)$, the admissibility, where $\emptyset$ is the empty word in $V(G) \cup FP(G)$. The binary operation is the admissibility (or connectedness) of elements in $V(G)$ $\cup\ FP(G)$ defined by

$$(w_1, w_2) \longmapsto \begin{cases} w_1 w_2 & \text{if } w_1 w_2 \in V(G) \cup FP(G) \\ \emptyset & \text{otherwise.} \end{cases}$$

The main purpose of this paper is to introduce the measures on a finite directed graph. Define the diagram map $\delta : \mathbb{F}^+(G) \to \mathbb{F}^+(G)$ by $w \mapsto \delta_w$, where the diagram $\delta_w$ of $w$ is nothing but the graphical image of $w$ in $\mathbb{R}^2$, on the given graph $G$. The image $D(G)$ of $\delta$ is a subset and it also has its admissibility, as binary operation, inherited by that of $\mathbb{F}^+(G)$. i.e., $\delta_w\ \delta_{w'} = \delta_{ww'}$. Thus the diagram set $D(G)$ with admissibility is again an algebraic structure.

In Chapter 1, we will define the measure $\mu_G$ on $D(G)$. It is defined by $d \cup \Delta$, where $d$ is the degree measure on $V(G)$ and $\Delta$ is the diagram length measure on $D(G) \setminus V(G)$. Simply, $d$ is determined by degrees of vertices and $\Delta$ is determined by so-called the diagram length. i.e.,

$$d(S_V) = \sum_{v \in S_V} \frac{\deg(v)}{|V(G)|} \quad \text{and} \quad \Delta(S_{FP}) = \sum_{w \in S_{FP}} L(w),$$

where $\frac{\deg(v)}{|V(G)|} = \frac{\deg_{in}(v) + \deg_{out}(v)}{|V(G)|}$ and $L(w) = W(w) \cdot f(w)$, where $w = \delta_w \in D(G)$ and $W$ is the corresponding weighting map of $w$ and $f$ is the length function on $D_{FP}(G) \overset{def}{=} D(G) \setminus V(G)$. Here, the sets $S_V$ and $S_{FP}$ are subsets of $V(G)$ and $D_{FP}(G)$, respectively. So, the bounded positive measure $\mu_G$ is defined by

$$\mu_G(S) = d(S') + \Delta(S''), \text{ for all } S \subseteq D(G)$$

where $S' = V(G) \cap S$ and $S'' = FP(G) \cap S$. It is automatically defined that $\mu_G(\emptyset) = 0$.

For the given graph $G$, we can define the shadow $G^{-1}$ of it, as an opposite directed graph of $G$. Also, we can have a new finite directed graph $G^{\hat{}}$ with its vertex set $V(G^{\hat{}}) = V(G)$ and its edge set $E(G^{\hat{}}) = E(G) \cup E(G^{-1})$. It is called the shadowed graph of $G$. By regarding $G^{\hat{}}$ as a finite directed graph, we can construct the free semigroupoid $\mathbb{F}^+(G^{\hat{}})$ of $G^{\hat{}}$ and similar to the previous discussion, we can get the graph measure $\mu_{G^{\hat{}}} = d \cup \Delta^{\hat{}}$ on $D_r(G^{\hat{}})$, where $D_r(G^{\hat{}})$ is the reduced diagram set. Remark that the free semigroupoid $\mathbb{F}^+(G^{\hat{}})$ of the shadowed graph $G^{\hat{}}$ is much bigger than $\mathbb{F}^+(G) \cup \mathbb{F}^+(G^{-1})$. However, the graph measure $\mu_G$ (or $\mu_{G^{-1}}$) is the restricted measure on $D(G)$ (resp. $D(G^{-1})$) of $\mu_{G^{\hat{}}}$ on $D_r(G^{\hat{}})$. Here, the reduced diagram set $D_r(G^{\hat{}})$ is the diagram set $D(G^{\hat{}})$ with the reducing property making all loop finite paths $\delta_{w^{-1}w}$ and $\delta_{ww^{-1}}$ be the corresponding initial-and-terminal vertices of them. In Chapter 2, we will consider this shadowed graph measure $\mu_{G^{\hat{}}}$ of $G$. This measure is also a bounded positive measure. We can show that if the finite directed graphs $G_1$ and $G_2$ are graph-isomorphic, then the measure spaces $(G_1^{\hat{}}, \mu_{G_1^{\hat{}}})$ and $(G_2^{\hat{}}, \mu_{G_2^{\hat{}}})$ are equivalent.

In Chapter 3, we will construct the integrals on the given graph measure space $(G^{\hat{}}, \mu_{G^{\hat{}}})$. In Chapter 4, we observe the restricted measures of $\mu_{G^{\hat{}}}$, induced by the subgraphs of $G^{\hat{}}$.

Finally, in Chapter 5, we construct a new graph measure $\mu = d \cup \omega$ now on the free semigroupoid $\mathbb{F}^+(G^{\hat{}})$ of the shadowed graph $G^{\hat{}}$. The construction is very similar to $\mu_{G^{\hat{}}}$. However, this measure $\mu$ on $\mathbb{F}^+(G^{\hat{}})$ is just a locally bounded positive measure, in general. Here $d$ is the degree measure on $V(G^{\hat{}})$ defined before. The weighted length measure $\omega$ is defined similar to $\Delta^r$ as follows;

$$\omega(S) = \sum_{w \in S} W(\delta_w^r) f(\delta_w^r), \text{ for all } S \subseteq FP(G^{\hat{}}).$$

## 1. The Diagram Graph Measures

In this chapter, we will construct the diagram graph measure. Throughout this chapter, let $G$ be a finite directed graph. The free semigroupoid $\mathbb{F}^+(G)$ of $G$ is defined as in Introduction. We will determine the measure $\mu_G$ on $D(G)$, where $D(G)$ is the diagram set of $G$. The measure space $(D(G), P(D(G))\ \mu_G)$ is the main object of this chapter, where $P(X)$ means the power set of an arbitrary set $X$.

Let $G$ be a finite directed graph, with its vertex set $V(G)$ and its edge set $E(G)$. Let $v \in V(G)$ be a vertex. We say that the vertex $v$ has degree $m$ if it has $m$-incident edges and denote it by $\deg(v) = m$. There are two kinds of incident edges of $v$, depending on the direction of $G$. First, there are inner-incident edges $e$ such that $e = e\ v$. Second, there are outer-incident edges $e'$ such that $e' = v\ e'$. The inner-degree $\deg_{in}(v)$ of the vertex $v$ is defined by

$$\deg_{in}(v) = \left|\{e \in E(G) : e = ev\}\right|$$

and the outer-degree $\deg_{out}(v)$ of $v$ is defined by

$$\deg_{out}(v) = \left|\{e \in E(G) : e = ve\}\right|.$$

The degree $\deg(v)$ of $v$ is the sum of $\deg_{in}(v)$ and $\deg_{out}(v)$,

$$\deg(v) = \deg_{in}(v) + \deg_{out}(v).$$

If $v$ is isolated without the incident edges, then $\deg(v) = 0$. Since the given graph $G$ is finite,

$$\deg(v) < \infty, \quad \text{for all} \quad v \in V(G)$$

and

$$\sum_{v \in V(G)} \tfrac{\deg(v)}{|V(G)|} \leq |V(G)| \cdot |E(G)| < \infty.$$

**Definition 1.1.** *Let $G$ be the given finite directed graph and $\mathbb{F}^{+}(G)$, the corresponding free semigroupoid of $G$ and let $w \in \mathbb{F}^{+}(G)$. Define the diagram $\delta_w$ of $w$ by the graphical image of $w$ on the graph $G$. We say that the element $w$ in $\mathbb{F}^{+}(G)$ is basic if the diagram $\delta_w$ of it is $w$, itself, on $G$. Define the set $D_k(G)$ of all diagrams with length $k$ by*

$$D_k(G) = \{\delta_w : |\delta_w| = k\}, \quad \textit{for all} \quad k \in \mathbb{N}.$$

*Remark that, by regarding $\delta_w$ as a finite path, we can determine the length $|\delta_w|$ of the diagram $\delta_w$. The set $D(G)$ of all diagrams is defined by $D(G) = \{\emptyset\} \cup (\cup_{n=0}^{\infty} D_n(G))$, where*

$$D_0(G) \overset{def}{=} \{\delta_v : v \in V(G)\} = V(G).$$

*Denote $D(G) \setminus (V(G) \cup \{\emptyset\})$ by $D_{FP}(G)$. i.e., $D_{FP}(G) = D(G) \cap FP(G)$.*

Notice that there exists $N \in \mathbb{N}$ such that

$$D(G) = \cup_{n=0}^{N} D_n(G),$$

since $G$ is a finite graph.

Let's discuss about the differences between $FP(G)$ and $D_{FP}(G)$. Assume that $l \in E(G)$ is a loop-edge. Then it has its diagram $\delta_l = l$ on $G$, because the diagram $\delta_l$ of $l$ is $l$, itself on the given graph $G$. In fact, by definition, $\delta_e = e$, for all $e \in E(G)$. In other words, $E(G) = D_1(G) \subset D(G)$. So, as a finite path, the loop-edge $l$ is basic. However, the diagram $\delta_{l^2}$ of the finite path $l^2$ (with its length 2) is $l = \delta_l$ on $G$, because the finite path $l^2$ also has its graphical image $l$ on $G$. So, in this case, the diagram $\delta_{l^2}$ of $l^2$ is exactly same as the diagram $\delta_l$ of the basic finite path $l$. More generally, all diagrams $\delta_{l^k}$ of loop finite paths $l^k$, $k \in \mathbb{N} \setminus \{1\}$, are same as $l = \delta_l$ of the basic finite path $l$.

**Example 1.1.** *Let $e = v_1 e v_2 \in E(G)$, with $v_1 \neq v_2$ in $V(G)$ and let $l = v_2 l v_2 \in FP(G)$ be a loop finite path. Then the finite paths $e$ and $l$ are admissible. i.e., the finite path $el \in FP(G)$. Assume that there exists a basic loop finite path $w = v_2$ $w$ $v_2$ such that $l = w^k$, for some $k \in \mathbb{N}$, in $FP(G)$. Then the diagram $\delta_{el}$ of $el$ is same as the diagram $\delta_{ew}$, and the diagram $\delta_{ew}$ is identified with $ew$ on $G$.*

Assume that $v \in \mathbb{F}^+(G)$ is a vertex of $G$. Then the diagram $\delta_v$ of $v$ is always $v$ on $G$. Also, if $e \in \mathbb{F}^+(G)$ is an edge contained in $E(G)$, then the diagram $\delta_e$ of $e$ is $e$ on $G$. Now, we can define the diagram map

$$\delta : \mathbb{F}^+(G) \to D(G) \text{ by the map sending } w \text{ to } \delta_w,$$

which is a surjection from $\mathbb{F}^+(G)$ onto $D(G)$. If $w_1$ and $w_2$ have the same diagram on $G$, then $\delta_{w_1} = \delta_{w_2}$ on $G$. We can easily get that;

**Lemma 1.1.** *(1) If $v \in V(G)$, then $\delta_v = v$ on $G$.*
*(2) If $e \in E(G)$, then $\delta_e = e$ on $G$.*
*(3) An element $w \in \mathbb{F}^+(G)$ is basic if and only if $\delta_w = w$ on $G$.*
*(4) If $w \in \mathbb{F}^+(G)$ is not basic, then there exists a unique basic finite path $w_0 \in \mathbb{F}^+(G)$ such that $\delta_w = \delta_{w_0}$ on $G$.*

*Proof.* It suffices to prove the uniqueness in (4). The existence is guaranteed by (3). i.e., the diagram $\delta_w$ of $w$ in $D(G)$ is the basic element having the same diagram with that of $w$. i.e., $\delta_w = \delta_{\delta_w}$. Suppose that $w_1$ and $w_2$ are basic elements such that $\delta_{w_1} = \delta_w = \delta_{w_2}$. By the basicness of $w_1$ and $w_2$, we have $w_1 = \delta_{w_1} = \delta_{w_2} = w_2$. ∎

By definition of finite directed trees, $D(G) = \mathbb{F}^+(G)$, whenever $G$ is a tree.

**Proposition 1.2.** *Let $G$ be a finite directed tree. Then $\mathbb{F}^+(G) = D(G)$.* □

We will construct the measure $\mu_G$ on the free semigroupoid $D(G)$ of the finite directed graph $G$. First, consider the length function $f$ on $D(G)$. Define this map

$$f : D(G) \to \mathbb{R}$$

by

$$f(w) = k, \text{ for all } w \in D_k(G).$$

Remark that there exists $N \in \mathbb{N}$ such that $D(G) = \cup_{n=0}^{N} D_n(G)$, for any finite graph $G$. Moreover, since $G$ is finite $|D_k(G)| < \infty$, for all $k = 0, 1, ..., N$. So, we can define the following map $F : P(D(G)) \to \mathbb{N}$,

$$F(S) = \sum_{\delta \in S} f(\delta),$$

for all $S \in P(D(G))$, where $P(D(G))$ is the power set of $D(G)$. For instance, if $S = \{\delta_1, \delta_2\}$ and if $\delta_j \in D_{k_j}(G)$, for $j = 1, 2$, then

$$F(S) = k_1 + k_2 \in \mathbb{N}.$$

This map $F$ is called the diagram-length map.

**Lemma 1.3.** *Let $G$ be a finite directed graph and let $D(G)$ be the set of all diagrams on $G$. Then the map $F$ is bounded. i.e., $F(S) < \infty$, for all $S \in P(D(G))$.*

*Proof.* Let's take $S = D(G)$, the largest subset of $D(G)$. It is sufficient to show that $F(D(G)) < \infty$. Observe

$$F(D(G)) = F\left(\cup_{k=0}^{N} D_k(G)\right) = \sum_{k=0}^{N} F(D_k(G))$$

for some $N \in \mathbb{N}$, since $D_k(G)$'s are mutually disjoint

$$= 0 \cdot |V(G)| + \sum_{k=1}^{N} k\, |D_k(G)|$$

since $F(D_k(G)) = \sum_{\delta \in D_k(G)} f(\delta) = k \cdot |D_k(G)|$, for all $k = 0, 1, ..., N$,

$$< \infty,$$

since $G$ is finite and hence $D_k(G)$ has the bounded cardinality, for all $k = 1, ..., N$. ∎

Define the weighted diagram-length function $L$ on $D(G)$.

**Definition 1.2.** *Let $G$ be a finite directed graph and $\mathbb{F}^+(G)$, the corresponding free semigroupoid of $G$ and let $D(G)$ be the set of all diagrams on $G$. Let $F : P(D_{FP}(G)) \rightarrow \mathbb{N}$ be the diagram-length map defined above. Let $w = e_1 \ \dots \ e_k$ be a* basic *finite path with admissible edges $e_1$, ..., $e_k$, where $k \in \mathbb{N}$. Define the weighting $W_E : E(G) \rightarrow (0,\ 1)$ by $W_E(e) = r_e$, where $(0,\ 1)$ is the open interval in $\mathbb{R}$. More generally, define the weighting map $W : D_{FP}(G) \rightarrow (0,\ 1)$ by*

$$W(w) = \begin{cases} W_E(w) & \text{if } w \in E(G) \\ \\ \Pi_{j=1}^{k} W_E(e_j) & \text{if } w = e_1 ... e_k \in D_{FP}(G) \end{cases}$$

*Define the weighted diagram-length map (in short, WDL-map) $L : P\left(D_{FP}(G)\right) \rightarrow (0,\ 1)$ by*

$$L(S) \overset{def}{=} \sum_{w \in S} W(w) f(w), \text{ for all } S \in P\left(D_{FP}(G)\right).$$

*The unweighted diagram-length map (in short, UWDL-map) $L_u$ on $P(D_{FP}(G))$ is defined by $L_u\ (S) = F\ (S)$, for all $S$ in $P\left(D_{FP}(G)\right)$.*

By definition, we can get that;

**Lemma 1.4.** *Let $G$ be a finite directed graph and the map $L : P\left(D_{FP}(G)\right) \rightarrow \mathbb{R}$ be the WDL-map. Then it is bounded. The UWDL-map $L_u$ is also bounded.*

*Proof.* It suffices to show that $L\left(D_{FP}(G)\right)$ is bounded. By definition, we have that

$$L\left(D_{FP}(G)\right) = \sum_{w \in D_{FP}(G)} W(w) f(w) < \infty,$$

since the diagram set $D(G)$ has a finite cardinality and so does $D_{FP}(G)$. Similarly, we can show that

$$L_u\ \left(D_{FP}(G)\right) = F\ \left(D_{FP}(G))\right) = F\left(\cup_{k=1}^{N} D_k(G)\right) < \infty.$$

∎

Now, we will define the measure $\mu_G$ on $D(G)$.

**Definition 1.3.** *Let $G$ be a finite directed graph and let $D(G)$ be the corresponding diagram set of $G$. Define the degree measure $d$ on the vertex set $V(G)$ by*

$$d(S) = \sum_{v \in S} \frac{\deg(v)}{|V(G)|}, \quad \text{for all} \quad S \subseteq V(G).$$

*Also, define the weighted measure $\Delta$ on the finite path set $D(G) \cap FP(G)$ by*

$$\Delta(S) = L(S), \text{ for all } S \subseteq D_{FP}(G).$$

*The unweighted measure* $\Delta_u$ *on* $FP(G)$ *by*

$$\Delta_u(S) = L_u(S), \text{ for all } S \subseteq D_{FP}(G).$$

*Finally, we have the (weighted) graph measure (or G-measure)* $\mu_G$ *on* $D(G)$*, denoted by* $d \cup \Delta$*, defined by*

$$\mu_G(S) = d(S_V) + \Delta(S_{FP}), \text{ for all } S \subseteq D(G),$$

*where* $S_V = S \cap V(G)$ *and* $S_{FP} = S \cap D_{FP}(G)$*. Similarly, the unweighted G-measure* $\mu_G^u$ *is defined by* $d \cup \Delta_u$*.*

The $G$-measure $\mu_G$ and $\mu_G^u$ are indeed bounded measures on $D(G)$. Instead of using $P(D(G))$, we will use the quotient set $P\left(\mathbb{F}^+(G)\right) / \mathcal{R}$.

**Theorem 1.5.** *The G-measure* $\mu_G$ *is a bounded positive measure on* $P\left(D(G)\right)$*. The unweighted G-measure* $\mu_G^u$ *is also a bounded positive measure on* $P\left(D(G)\right)$*.*

*Proof.* To show that $\mu_G = d \cup \Delta$ is a measure on $D(G)$, we will check the followings;

(i) $\mu_G(S) \geq 0$, for all $S \subseteq D(G)$. If $S$ is a subset of $D(G)$, then there always is a separation $S_V$ and $S_{FP}$ of $S$, where $S_V = V(G) \cap S$ and $S_{FP} = D_{FP}(G) \cap S$, such that $S_V \cup S_{FP} = S$ and $S_V \cap S_{FP} = \varnothing$. So,

$$\mu_G(S) = d(S_V) + \Delta(S_{FP}) = \sum_{v \in S_V} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{FP}} W(w) f(w) \geq 0.$$

(ii) For any $S \subseteq D(G)$, $\mu_G(S) < \infty$. We need to show that $\mu_G\left(D(G)\right) < \infty$. We have that

$$\mu_G\left(D(G)\right) = d\left(V(G)\right) + \Delta\left(D_{FP}(G)\right).$$

By the previous lemma, $\Delta\left(D_{FP}(G)\right) < \infty$. Also,

$$d\left(V(G)\right) = \sum_{v \in V(G)} \frac{\deg(v)}{|V(G)|} \leq |V(G)| \cdot |E(G)| < \infty,$$

since $G$ is a finite graph. Therefore, $\mu_G\left(D(G)\right) < \infty$.

(iii) Let $(S_n)_{n=1}^{\infty}$ be a mutually disjoint sequence of subsets of $D(G)$. Then

$$\mu_G\left(\cup_{n=1}^{\infty} S_n\right) = \mu_G\left(\left(\cup_{n=1}^{\infty} S_{n,V}\right) \cup \left(\cup_{n=1}^{\infty} S_{n,FP}\right)\right)$$

where $S_{n,V} = V(G) \cap S_n$ and $S_{n,FP} = D_{FP}(G) \cap S_n$, for all $n \in \mathbb{N}$

$$= d\left(\cup_{n=1}^{\infty} S_{n,V}\right) + \Delta\left(\cup_{n=1}^{\infty} S_{n,FP}\right)$$

by the definition of $\mu_G = d \cup \Delta$

$$= \sum_{n=1}^{\infty} d\left(S_{n,V}\right) + \sum_{n=1}^{\infty} \Delta\left(S_{n,FP}\right)$$

by the definition of $d$ and $\Delta$ and by the disjointness of $\{S_{n,V}\}_{n=1}^{\infty}$ and $\{S_{n,FP}\}_{n=1}^{\infty}$

$$= \sum_{n=1}^{\infty} \mu_G(S_n).$$

By (i) and (iii), the $G$-measure $\mu_G = d \cup \Delta$ is a measure on $D(G)$, and by (ii), it is a bounded measure on $D(G)$. ∎

**Definition 1.4.** *Let $G$ be a finite directed graph. The triple $(D(G), P(D(G)), \mu_G)$ is called the graph measure space (in short, $G$-measure space). For convenience, the triple is denoted by $(G, \mu_G)$.*

**Proposition 1.6.** *Let $(G, \mu_G)$ be a $G$-measure space. Then the $G$-measure $\mu_G$ is atomic, in the sense that, for all $S \subseteq D(G) \setminus \{\emptyset\}$, $0 < \mu_G(S) < \infty$.* □

## 2. Reduced Diagram Graph Measures

Throughout this chapter, let $G$ be a finite directed graph and let $\mathbb{F}^+(G)$ be the corresponding free semigroupoid. In this chapter, we will construct the shadowed graph $G^{\hat{}}$ of the given graph $G$, as a new finite directed graph. By constructing the free semigroupoid $\mathbb{F}^+(G^{\hat{}})$ of the shadowed graph $G^{\hat{}}$, we can define the free groupoid $\mathbb{F}^+(G^{\hat{}})$ of the given graph $G$. Similar to the previous chapter, we can have the $G^{\hat{}}$-measure space

$$\left(G^{\hat{}},\ \mu_{G^{\hat{}}}\right) = \left(D_r(G^{\hat{}}),\ P\left(D_r(G^{\hat{}})\right),\ \mu_{G^{\hat{}}}\right).$$

But the graph measure $\mu_{G^{\hat{}}}$ is defined slightly differently. To define this $G^{\hat{}}$-measure $\mu_{G^{\hat{}}} = d \cup \Delta^{\hat{}}$, (in particular, to define $\Delta^{\hat{}}$), we will use the reduced diagrams $D_r(G^{\hat{}})$, instead of using the diagrams $D(G^{\hat{}})$.

## 2.1. **Shadowed Graphs.**

Let $G$ be the given finite directed graph and $\mathbb{F}^{+}(G)$, the corresponding free semigroupoid.

**Definition 2.1.** *Let $G$ be the given graph. Then the shadow $G^{-1}$ is defined by a graph with*

$$V(G^{-1}) = V(G) \quad and \quad E(G^{-1}) = \{e^{-1} : e \in E(G)\},$$

*where $e^{-1}$ is the opposite directed edge of $e$. i.e., the shadow $G^{-1}$ is the opposite directed graph of $G$.*

Let $e = v_1 e v_2$, with $v_1,\, v_2 \in V(G)$. Then the opposite directed edge $e^{-1}$ of $e$ satisfies that $e^{-1} = v_2\ e^{-1}\ v_1$. Notice that the admissibility on $G^{-1}$ is preserved oppositely by that of $G$. i.e., the finite path $w^{-1} = e_1^{-1}\ ...\ e_k^{-1}$ is in $FP(G^{-1})$ if and only if $w = e_k\ ...\ e_1$ is in $FP(G)$. More generally, the element $w^{-1} = w_1^{-1}\ ...\ w_k^{-1}$ is in $\mathbb{F}^{+}(G^{-1})$ if and only if $w = w_k\ ...\ w_1$ is in $\mathbb{F}^{+}(G)$.

**Proposition 2.1.** $\left|\mathbb{F}^{+}(G^{-1})\right| = \left|\mathbb{F}^{+}(G)\right|.$

*Proof.* By definition, the free semigroupid $\mathbb{F}^{+}(G^{-1})$ of $G^{-1}$ is $\{\emptyset\} \cup V(G) \cup FP(G^{-1})$. So, it suffices to show that

$$\left|FP(G^{-1})\right| = \left|FP(G)\right|.$$

There exists a natural map from $FP(G)$ onto $FP(G^{-1})$ defined by $w \mapsto w^{-1}$. Since the admissibility on $G^{-1}$ is preserved oppositely by that of $G$ and since we can regard $G$ as $(G^{-1})^{-1}$, the map $w \mapsto w^{-1}$ is bijective. So, the above equality holds. ∎

Since $\left|\delta\left(\mathbb{F}^{+}(G^{-1})\right)\right| = \left|\delta\left(\mathbb{F}^{+}(G)\right)\right|,$ where $\delta$ is the diagram map, we can get that;

**Corollary 2.2.** $\left|D(G^{-1})\right| = \left|D(G)\right|.$ $\square$

By the previous two results, we can get that;

**Theorem 2.3.** *The graph measure spaces $(G,\, \mu_G)$ and $(G^{-1},\, \mu_{G^{-1}})$ are equivalent.* $\square$

Now, let's define the shadowed graph $\hat{G}$ of $G$.

**Definition 2.2.** *Let $G$ be given as above and let $G^{-1}$ be the shadow of $G$. Define the shadowed graph $G^{\hat{}}$ by the directed graph with*

$$V(G^{\hat{}}) = V(G) = V(G^{-1}) \quad and \quad E(G^{\hat{}}) = E(G) \cup E(G^{-1}).$$

Notice that $\mathbb{F}^{+}(G^{\hat{}}) \neq \mathbb{F}^{+}(G) \cup \mathbb{F}^{+}(G^{-1})$, since we can also take a mixed finite path $w_1^{i_1}\ w_2^{i_2}\ ...\ w_m^{i_m}$, where $(i_1,\ ...,\ i_m) \in \{1,\ -1\}^m$ is a mixed $m$-tuple of 1 and $-1$, in $\mathbb{F}^{+}(G^{\hat{}})$. Such mixed finite paths are not contained in $\mathbb{F}^{+}(G) \cup \mathbb{F}^{+}(G^{-1})$. Therefore, in general,

$$\mathbb{F}^{+}(G^{\hat{}}) \supsetneqq \left(\mathbb{F}^{+}(G) \cup \mathbb{F}^{+}(G^{-1})\right).$$

## 2.2. **Reduced Diagrams on the Shadowed Graph.**

Let $G$ be the given finite directed graph and $G^{-1}$, the shadow of $G$, and let $G^{\hat{}}$ be the shadowed graph of $G$. Like Chapter 1, we can define the diagram $D(G^{\hat{}})$ of $G^{\hat{}}$, as the image of the diagram map $\delta : \mathbb{F}^{+}(G^{\hat{}}) \rightarrow D(G^{\hat{}})$ defined by $w \mapsto \delta_w$. On this diagram set $D(G^{\hat{}})$, we will give the reducing process and we will define the reduced diagram set $D_r(G^{\hat{}})$ of $G^{\hat{}}$. This reduced diagram set $D_r(G^{\hat{}})$ is a groupoid, under the admissibility inherited by that of $D(G)$, with the reducing property.

**Definition 2.3.** *Let $G$, $G^{-1}$ and $G^{\hat{}}$ be given as above and let $\mathbb{F}^{+}(G^{\hat{}})$ be the free semigroupoid of $G^{\hat{}}$. Let $\delta : \mathbb{F}^{+}(G^{\hat{}}) \rightarrow D(G^{\hat{}})$ be the diagram map defined in Chapter 1, for the finite directed graph $G^{\hat{}}$. Define the reduced diagram map $\delta^r$ : $\mathbb{F}^{+}(G^{\hat{}}) \rightarrow D(G^{\hat{}})$ by*

$$w \longmapsto \delta_w^r = \delta_w,\ \forall\ w \in \mathbb{F}^{+}(G) \cup \mathbb{F}^{+}(G^{-1})$$

*and*

$$w^{-1}w \mapsto \delta_{w^{-1}w}^r = \delta_v \quad and \quad ww^{-1} \mapsto \delta_{w^{-1}w}^r = \delta_{v'}, \tag{2.1}$$

*for all $w = v'wv \in FP(G)$ and $w^{-1} = vw^{-1}v' \in FP(G^{-1})$, with $v,\ v' \in V(G)$, and*

$$w_1^{i_1}w_2^{i_2}...w_n^{i_n} \longmapsto \delta_{w_1^{i_1}w_2^{i_2}...w_n^{i_n}}^r = \delta_{\delta_{w_1^{i_1}...w_n^{i_n}}}^r,$$

*for the $n$-tuple $(i_1,\ ...,\ i_n) \in \{1,\ -1\}^n$, for all $n \in \mathbb{N}$. The image $\delta_w^r$ of $w$ in $\mathbb{F}^{+}(G)$ is called the reduced diagram of $w$. Define the set $D_r(G)$ of all reduced diagrams by the image $\delta^r\left(\mathbb{F}^{+}(G^{\hat{}})\right)$ of $\delta^r$. i.e.,*

$$D_r(G) \overset{def}{=} \delta^r\left(\mathbb{F}^{+}(G^{\hat{}})\right) = \delta^r\left(\delta(\mathbb{F}^{+}(G^{\hat{}}))\right) = \delta^r\left(D(G^{\hat{}})\right).$$

2.3. **Reduced Diagram Graph Measure Spaces.**

Let $G$, $G^{-1}$ and $G^{\hat{}}$ be given as above. Then we can define the reduced diagram graph measure $\mu_{G^{\hat{}}}$ by $\mu_{G^{\hat{}}} = d \cup \Delta^{\hat{}}$.

**Definition 2.4.** *Let $G^{\hat{}}$ be the shadowed graph of a finite directed graph $G$ and let $\Delta^{\hat{}}$ be the reduced diagram length measure defined by the diagram length measure on $D_r^{FP}(G^{\hat{}}) \overset{def}{=} D_r(G^{\hat{}}) \cap FP(G^{\hat{}})$. Define the reduced diagram graph measure $\mu_{G^{\hat{}}}$ by $d \cup \Delta^{\hat{}}$. i.e.,*

$$\mu_{G^{\hat{}}}(S) = d\left(S_V\right) + \Delta^{\hat{}}\left(S_{FP}\right),$$

*for all $S \subseteq D_r(G^{\hat{}})$, where $S_V = S \cap V(G^{\hat{}})$ and $S_{FP} = S \cap D_r^{FP}(G^{\hat{}})$. The triple $\left(D_r(G^{\hat{}}),\ P\left(D_r(G^{\hat{}})\right),\ \mu_{G^{\hat{}}}\right)$ is called the reduced diagram graph measure space of the graph $G$ (in short, $G^{\hat{}}$-measure space of $G$), denoted by $(G^{\hat{}},\ \mu_{G^{\hat{}}})$.*

Then, similar to Chapter 1, we can get that;

**Theorem 2.4.** *Let $(G^{\hat{}},\ \mu_{G^{\hat{}}})$ be the $G^{\hat{}}$-measure space of $G$. Then $\mu_{G^{\hat{}}}$ is bounded positive measure on $D_r(G^{\hat{}})$. In particular, this measure is atomic, in the sense that, all subsets $S$ in $P\left(D_r(G^{\hat{}})\right)$, which is not $\{\emptyset\}$, have bounded measures.* $\square$

The key idea to prove the above theorem is to understand the reduced diagram set $D_r(G^{\hat{}})$ has finite cardinality. We can use the same techniques used in Chapter 1.By definition, the following proposition is easily proved.

**Proposition 2.5.** *Let $\mu_G$ and $\mu_{G^{-1}}$ be the $G$-measure on $D(G)$ and the $G^{-1}$-measure on $D(G^{-1})$, respectively. Then the $G^{\hat{}}$-measure $\mu_{G^{\hat{}}}$ satisfies that $\mu_{G^{\hat{}}} = \mu_G$ on $D(G)$ and $\mu_{G^{\hat{}}} = \mu_{G^{-1}}$ on $D(G^{-1})$.* $\square$

Let $G_1$ and $G_2$ be finite directed graph and $\hat{G_1}$ and $\hat{G_2}$, the shadowed graphs of $G_1$ and $G_2$, respectively. Assume that $G_1$ and $G_2$ are graph-isomorphic. i.e., there exists a graph-isomorphism $\varphi : G_1 \to G_2$ such that (i) $\varphi$ is the bijection from $V(G_1)$ onto $V(G_2)$, (ii) $\varphi$ is the bijection from $E(G_1)$ onto $E(G_2)$, (iii) for $v$, $v'$ $\in V(G_1)$ such that $e = v\ e\ v'$ in $E(G_1)$, $\varphi(e) = \varphi(v)\ \varphi(e)\ \varphi(v')$. i.e., $\varphi$ preserves the admissibility. Then the $\hat{G_1}$-measure (resp. $G_1$-measure) and $\hat{G_2}$-measure (resp. $G_2$-measure) are equivalent.

**Theorem 2.6.** *Let $G_1$ and $G_2$ be graph-isomorphic finite directed graphs. Then the shadowed graph measure spaces $(\hat{G_1},\ \mu_{\hat{G_1}})$ and $(\hat{G_2},\ \mu_{\hat{G_2}})$ are an equivalent measure spaces. In particular, the graph measure spaces $(G_1,\ \mu_{G_1})$ and $(G_2,\ \mu_{G_2})$ are also equivalent measure spaces.* $\square$

By the definition of the graph-isomorphisms, we can easily prove the above theorem.The main idea is that $D_r(\hat{G_1})$ (resp. $D(G_1)$) and $D_r(\hat{G_2})$ (resp. $D(G_2)$) are reduced-diagram-isomorphic (resp. diagram-isomorphic). In fact, we can have that

$G_1$ and $G_2$ are graph-isomorphic

$\Longrightarrow \hat{G_1}$ and $\hat{G_2}$ are graph-isomorphic

$\Longrightarrow \mathbb{F}^+(\hat{G_1})$ and $\mathbb{F}^+(\hat{G_2})$ are free–semigroupoid-isomorphic

$\Longrightarrow D(\hat{G_1})$ and $D(\hat{G_2})$ are diagram-isomorphic

$\Longrightarrow D_r(\hat{G_1})$ and $D_r(\hat{G_2})$ are reduced-diagram-isomorphic

$\Longrightarrow (\hat{G_1}, \mu_{\hat{G_1}})$ and $(\hat{G_2}, \mu_{\hat{G_2}})$ are equivalent.

In [8], we discussed about it more in detail. In fact, under certain condition, the converses also hold true. However, without the condition, we cannot guarantee that the converses hold true.

## 3. Measure Theory on Graphs

In this chapter, we will consider Measure theory on the given finite directed graph $G$, with respect to the reduced diagram graph measure space $(\hat{G}, \mu_{\hat{G}})$. Of course, we can consider Measure theory with respect to the graph measure space $(G, \mu_G)$. But Calculus on $(G, \mu_G)$ can be regarded as the restricted Calculus on $(\hat{G}, \mu_{\hat{G}})$. So, we will concentrate on observing Measure theory on $(\hat{G}, \mu_{\hat{G}})$.

### 3.1. $\hat{G}$-Measurable Functions.

Let $G$ be the given finite directed graph and $\hat{G}$, the shadowed graph and let $\mathbb{F}^+(\hat{G})$ be the free semigroupoid of $\hat{G}$, and $(\hat{G}, \mu_{\hat{G}})$, the $\hat{G}$-measure space. All simple functions $g$ are defined by

$$g = \sum_{n=1}^{N} a_n 1_{S_n}, \quad \text{for} \quad a_j \in \mathbb{R}, \tag{3.1}$$

where $S_1, ..., S_N$ are subsets of $D_r(\hat{G})$ and where

$$1_{S_j}(w) = \begin{cases} 1 & \text{if } w \in S_j \\ 0 & \text{otherwise,} \end{cases}$$

which are called the characteristic functions of $S_j$, for all $j = 1, ..., N$. All characteristic functions are $G^{\hat{}}$-measurable. And all $G^{\hat{}}$-measurable functions are approximated by simple functions.

**Example 3.1.** *Let $w \in D_r(G^{\hat{}})$. Then this element $w$ acts as one of the $G^{\hat{}}$-measurable function from $D_r(G^{\hat{}})$ to $\mathbb{R}$. Note that $w$ acts on $D_r(G^{\hat{}})$, as the left multiplication or the right multiplication. So, we have two different functions $g_l^w$ and $g_r^w$ on $D_r(G^{\hat{}})$ such that*

$$g_l^w = 1_{\delta^r(S_l^w)} \quad \text{and} \quad g_r^w = 1_{\delta^r(S_r^w)},$$

*for all $w' \in D_r(G^{\hat{}})$, where*

$$S_l^w = \{w'' \in \mathbb{F}^+(G^{\hat{}}) : ww'' \in \mathbb{F}^+(G^{\hat{}})\}$$

*and*

$$S_r^w = \{w''' \in \mathbb{F}^+(G^{\hat{}}) : w'''w \in \mathbb{F}^+(G^{\hat{}})\}.$$

*Therefore, the element $w$ act as a $G^{\hat{}}$-measurable function $g_w$ on $D_r(G^{\hat{}})$ defined by*

$$g_w = 1_{\delta^r(S_l^w) \cup \delta^r(S_r^w)}.$$

### 3.2. **Integration on Graphs.**

In this section, we will define the integrals of the given $G^{\hat{}}$-measurable functions, with respect to the $G^{\hat{}}$-measure $\mu_{G^{\hat{}}}$. First, let $g$ be a simple function given in (3.1). Then the integral $I_G(g)$ of $g$ with respect to $\mu_{G^{\hat{}}}$ is defined by

$$(3.2) \qquad I_G(g) \overset{denote}{=} \int_{G^{\hat{}}} g \; d\mu_{G^{\hat{}}} \overset{def}{=} \sum_{n=1}^{N} a_n \; \mu_{G^{\hat{}}}(S_n).$$

Since $\mu_{G^{\hat{}}} = d \cup \Delta^{\hat{}}$, the definition (3.2) can be rewritten as

$$\begin{aligned} I_G(g) &= \sum_{n=1}^{N} a_n \mu_{G^{\hat{}}}(S_n) = \sum_{n=1}^{N} a_n \left( d(S_{n,V}) + \Delta^{\hat{}}(S_{n,FP}) \right) \\ &= \sum_{n=1}^{N} a_n \left( \sum_{v \in S_{n,V}} \frac{\deg(v)}{|V(G)|} + \sum_{\delta_w^r \in S_{n,FP}} W(\delta_w^r) f(\delta_w^r) \right). \end{aligned}$$

**Proposition 3.1.** *Let $g_1 = \sum_{j=1}^{n} 1_{S_j}$ and $g_2 = \sum_{i=1}^{m} 1_{T_i}$ be simple functions, where $S_j$'s and $T_i$'s are subsets of $D_r(G^{\hat{}})$. Suppose that $S_j$'s are mutually disjoint*

*and also $T_i$'s are mutually disjoint. If $\cup_{j=1}^{n} S_j = \cup_{i=1}^{m} T_i$ in $D_r(G^{\hat{}})$, then $I_G(g_1) = I_G(g_2)$.*

*Proof.* Observe that

$$I_G(g_1) = \sum_{j=1}^{n} \left( \sum_{v \in S_{j,V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{j,FP}} W(w) f(w) \right) \tag{3.3}$$

and

$$I_G(g_2) = \sum_{i=1}^{m} \left( \sum_{v \in T_{i,V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in T_{i,FP}} W(w) f(w) \right). \tag{3.4}$$

By the assumption, $\cup_{j=1}^{n} S_j$ and $\cup_{i=1}^{m} T_i$ coincide in $D_r(G^{\hat{}})$. Therefore, we have that

$$\begin{aligned} \cup_{j=1}^{n} S_j &= \left(\cup_{j=1}^{n} S_j\right)_V + \left(\cup_{j=1}^{n} S_j\right)_{FP} \\ &= \left(\cup_{i=1}^{m} T_i\right)_V + \left(\cup_{i=1}^{m} T_i\right)_{FP} \\ &= \cup_{i=1}^{m} T_i. \end{aligned} \tag{3.5}$$

Since $\{S_j\}_{j=1}^{n}$ and $\{T_i\}_{i=1}^{m}$ are disjoint families of $P\left(D_r(G^{\hat{}})\right)$, the relation (3.5) explains

$$\begin{aligned} \left(\cup_{j=1}^{n} S_j\right)_V &= \cup_{j=1}^{n} S_{j,V} \\ &= \cup_{i=1}^{m} T_{i,V} \\ &= \left(\cup_{i=1}^{m} T_i\right)_V. \end{aligned} \tag{3.6}$$

Similarly, we have that

$$\left(\cup_{j=1}^{n} S_j\right)_{FP} = \cup_{j=1}^{n} S_{j,FP} = \cup_{i=1}^{m} T_{i,FP} = \left(\cup_{i=1}^{m} T_i\right)_{FP}. \tag{3.7}$$

Now, notice that

$$\sum_{j=1}^{n} \sum_{v \in S_{j,V}} \frac{\deg(v)}{|V(G)|} = \sum_{v \in \cup_{j=1}^{n} S_{j,V}} \frac{\deg(v)}{|V(G)|}$$

and

$$\sum_{i=1}^{m} \sum_{v \in T_{i,V}} \frac{\deg(v)}{|V(G)|} = \sum_{v \in \cup_{i=1}^{m} T_{i,V}} \frac{\deg(v)}{|V(G)|},$$

by the mutually disjointness on $\{S_{j,V}\}_{j=1}^{n}$ and $\{T_{i,V}\}_{i=1}^{m}$. Similarly, by the mutually disjointness of $\{S_{j,FP}\}_{j=1}^{n}$ and $\{T_{i,FP}\}_{i=1}^{m}$ and by (3.7),

$$\sum_{j=1}^{n} \sum_{w \in S_{j,FP}} L(\delta_w^r) = \sum_{w \in \cup_{j=1}^{n} S_{j,FP}} L(w)$$

and

$$\sum_{i=1}^{m} \sum_{w \in T_{i,FP}} L(\delta_w^r) = \sum_{w \in \cup_{i=1}^{m} T_{i,FP}} L(w).$$

Therefore, we can conclude that $I_G(g_1) = I_G(g_2)$, by (3.5). ∎

It is easy to check that, if $g_j = \sum_{k=1}^{n} a_{j,k} 1_{S_{j,k}}$, for $j = 1$, 2, then we can have

$$I_G(g_1 + g_2) = I_G(g_1) + I_G(g_2) \tag{3.8}$$

$$I_G(c \cdot g_j) = c \cdot I_G(g_j), \text{ for all } j = 1, 2. \tag{3.9}$$

Now, let $1_S$ and $1_T$ be the characteristic functions, where $S \neq T$ in $P\left(D_r(G^{\hat{}})\right)$. Then we have that $1_S \cdot 1_T = 1_{S \cap T}$.

**Proposition 3.2.** *Let $g_j$ be given as above, for $j = 1$, 2. Then*

$$I_G(g_1 g_2) = \sum_{k,i=1}^{n} (a_{1,k} a_{2,i}) \cdot \left( \sum_{v \in S_{(1,k),V} \cap S_{(2,i),V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{(1,k),FP} \cap S_{(2,i),FP}} L^{\hat{}}(w) \right).$$

*Proof.* Observe that

$$g_1 g_2 = \left(\sum_{k=1}^{n} a_{1,k} 1_{S_{1,k}}\right) \left(\sum_{i=1}^{n} a_{2,i} 1_{S_{2,i}}\right)$$

$$= \sum_{k,i=1}^{n} a_{1,k} a_{2,i} \left(1_{S_{1,k}} \cdot 1_{S_{2,i}}\right) = \sum_{k,i=1}^{n} a_{1,k} a_{2,i} 1_{S_{1,k} \cap S_{2,i}}$$

So, we have that

$$I_G(g_1 g_2) = \sum_{k,i=1}^{n} a_{1,k} a_{2,i} \mu_{G^{\hat{}}} \left(S_{1,k} \cap S_{2,i}\right). \tag{3.10}$$

Consider

$$\mu_{G^{\hat{}}} \left(S_{1,k} \cap S_{2,i}\right) = d\left((S_{1,k} \cap S_{2,i})_V\right) + \Delta^{\hat{}} \left((S_{1,k} \cap S_{2,i})_{FP}\right)$$

$$= d\left(S_{(1,k),V} \cap S_{(2,i),V}\right) + \Delta^{\hat{}} \left(S_{(1,k),FP} \cap S_{(2,k),FP}\right)$$

$$= \sum_{v \in S_{(1,k),V} \cap S_{(2,i),V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{(1,k),FP} \cap S_{(2,i),FP})} L^{\hat{}}(w).$$

∎

Suppose that $g_1$ and $g_2$ are given as above and assume that the families $\{S_{1,k}\}_{k=1}^n$ and $\{S_{2,i}\}_{i=1}^n$ are disjoint family in $D_r(G^{\hat{}})$. i.e.,

$$(\cup_{k=1}^n S_{1,k}) \cap (\cup_{i=1}^n S_{2,i}) = \varnothing.$$

Then $I(g_1 g_2) = 0$.

From now, similar to the classical measure theory, we can define the integration of $G^{\hat{}}$-measurable functions.

**Definition 3.1.** *Suppose that $f$ is a positive $G^{\hat{}}$-measurable function.. i.e., $f = f^+$. Then we define the $G$-integral $I_G(f)$ of $f$ with respect to $\mu_{G^{\hat{}}}$ by*

$$I_G(f) \overset{denote}{=} \int_{G^{\hat{}}} f \; d\mu_{G^{\hat{}}} = \sup_{g \leq f, \; g \; is \; simple} I_G(g).$$

*Suppose that $f$ is $G^{\hat{}}$-measurable. Then, since $f = f^+ - f^-$, we can define the $G$-integral of $f$ by*

$$I_G(f) = I_G(f^+) - I_G(f^-).$$

Therefore, the positive $G^{\hat{}}$-measurable function $|f|$ has its graph integral,

$$I_G\left(|f|\right) = I_G(f^+) + I_G(f^-).$$

If the $G^{\hat{}}$-measurable function $f$ satisfies $I_G(|f|) < \infty$, then we say that this map $f$ is bounded.

Let's observe more $G^{\hat{}}$-measurable functions. If $g$ is a $G^{\hat{}}$-measurable function on $D_r(G^{\hat{}})$, then the support of $g$ is denoted by $D_r(G^{\hat{}} : g)$. First, consider the polynomial $g_1(x) = g_x$ on $D_r(G^{\hat{}})$, where $g_x = 1_{\delta^r(S_l^x) \cup \delta^r(S_r^x)}$. This polynomial $g_1$ maps all $w$ in $\mathbb{F}^+(G^{\hat{}})$ to $g_w$. Then, we can have the $G^{\hat{}}$-measurable function $g_1$ on $D_r(G^{\hat{}})$ defined by $g_1(w) = 1_{\delta^r(S_l^w) \cup \delta^r(S_r^w)}$, for all $w \in D_r(G)$.

$$I_G(g_1) = \sum_{w \in D_r(G^{\hat{}})} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right), \tag{3.11}$$

by the previous proposition. Notice that the support $D_r(G^{\hat{}} : g_1)$ of $g_1$ is identified with $D_r(G^{\hat{}})$.

Now, let $g_2(x) = g_{\delta_{x^2}^r}$ on $D_r(G^{\hat{}})$. This polynomial $g_2$ has its support

$$D_r(G^{\hat{}} : g_2) = \{w \in D_r(G^{\hat{}}) : w \in V(G) \text{ or } w \text{ is a loop finite path}\}.$$

The above support $D_r(G^{\hat{}} : g_2)$ is determined by the fact that $g_2(v) = g_{v^2} = g_v$, for all $v \in V(G)$, and $g_2(l) = g_{\delta^r_{l^2}} = g_l$, for all loop finite paths $l$ in $D_r(G)$. Notice that the loop finite path in $D_r(G^{\hat{}})$ is the loop which is neither of the form $w^{-1}\ w$ nor of the form $w\ w^{-1}$, by the reducing process (2.1). Assume now that $w = v_1\ w\ v_2$ is a non-loop finite path in $D_r(G^{\hat{}})$ with $v_1 \neq v_2$ in $V(G^{\hat{}})$. Then $w^2 = (v_1\ w\ v_2)\ (v_1\ w\ v_2) = \emptyset$, in $D_r(G^{\hat{}})$. Therefore, we can get that

$$(3.12)\quad I_G(g_2) = \int_{G^{\hat{}}} g_2\, d\mu_{G^{\hat{}}} = \sum_{w \in D_r(G^{\hat{}} : g_2)} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right).$$

In general, we can get that;

**Proposition 3.3.** *Let $g_n(x) = g_{x^n}$ be the monomial on $D_r(G^{\hat{}})$. Then $I_G(g_n) = I_G(g_2)$, for all $n \in \mathbb{N} \setminus \{1\}$, where $I_G(g_2)$ is given in (3.12).*

*Proof.* It suffices to show that the support $D_r(G^{\hat{}} : g_n)$ of $g_n$ and the support $D_r(G : g_2)$ of $g_2$ coincide, for all $n \in \mathbb{N} \setminus \{1\}$. It is easy to check that if $w$ is in $\mathbb{F}^+(G^{\hat{}})$, then $w^n$ exists in $\mathbb{F}^+(G^{\hat{}})$ if and only if either $w$ is a vertex or $w$ is a loop finite path. So, the support of $g_n$ is

$$D_r(G^{\hat{}} : g_n) = V(G) \cup loop_r(G),$$

where $loop_r(G) \overset{def}{=} \{l \in D_r(G) : l \text{ is a loop finite path}\}$. Therefore, the support $D_r(G^{\hat{}} : g_n)$ is same as $D_r(G^{\hat{}} : g_2)$. Therefore,

$$\begin{aligned} I_G(g_n) &= \sum_{w \in D_r(G^{\hat{}} : g_2)} I_G(g_w) \\ &= \sum_{w \in D_r(G^{\hat{}} : g_2)} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right). \end{aligned}$$

∎

**Theorem 3.4.** *Let $g_p = \sum_{n=0}^{N} a_n g_n$ be a polynomial with $g_0 \equiv 1$, for $a_0, ..., a_N \in \mathbb{R}$. Then*

*(3.13)*

$$\begin{aligned} I_G(g_p) = \quad & a_0 \mu_{G^{\hat{}}}\left(D_r(G^{\hat{}})\right) + a_1 \left( \sum_{w \in D_r(G^{\hat{}})} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right) \right) \\ & + \sum_{k=2}^{N} a_k \left( \sum_{w \in V(G) \cup loop_r(G)} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right) \right), \end{aligned}$$

*where $loop_r(G) = \{l \in D_r(G^{\hat{}}) : l$ is a loop finite path$\}$.*

*Proof.* Let $g_p$ be the given polynomial on $\mathbb{F}^{+}(G^{\hat{}})$. Then the integral $I_G\left(g_p\right)$ of $g_p$ is determined by;

$$I_G\left(g_p\right) = \sum_{n=0}^{N} a_n I_G\left(g_n\right)$$

$$= a_0 I_G(1) + a_1 I_G\left(g_1\right) + \sum_{k=2}^{N} a_k I_G\left(g_k\right)$$

$$= a_0 \mu_{G^{\hat{}}}\left(D_r(G^{\hat{}})\right) + a_1 \left(\sum_{w \in D_r(G^{\hat{}})} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right)\right)$$

$$+ \sum_{k=2}^{N} a_k I_G\left(g_2\right)$$

since the constant function 1 has its support, $D_r(G^{\hat{}})$, and hence, since $I_G\left(g_n\right)$ $= I_G\left(g_2\right)$, for all $n \geq 2$, we can get that

$$= a_0 \mu_{G^{\hat{}}}\left(D_r(G)\right) + a_1 \left(\sum_{w \in D_r(G^{\hat{}})} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right)\right)$$

$$+ \sum_{k=2}^{N} a_k \left(\sum_{w \in V(G) \cup loop(G^{\hat{}})} \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right)\right),$$

because the support $D_r(G^{\hat{}} : g_2)$ of $g_2$ is the union of $V(G^{\hat{}})$ and $loop_r(G^{\hat{}})$, where $loop_r(G^{\hat{}})$ is the collection of all loop finite paths in $D_r(G^{\hat{}})$. ∎

**Corollary 3.5.** *Let $w = v_1 w v_2$ be a finite path in $D_r(G^{\hat{}})$ with $v_1 \neq v_2$ in $V(G)$, and let $g = \sum_{n=0}^{N} a_n g_w^n$, with $g_w^0 \overset{def}{=} 1$, where $a_0, ..., a_N \in \mathbb{R}$, and $g_w$ is $1_{\delta^r(S_l^w) \cup \delta^r(S_r^w)}$. Then*

$$I_G\left(g\right) = a_0 \mu_{G^{\hat{}}}\left(D_r(G^{\hat{}})\right) + a_1 \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right).$$

*Proof.* Since $w$ is a non-loop finite path, $w^k = \emptyset$, for all $k \in \mathbb{N} \setminus \{1\}$. Therefore, $S_l^{w^k} \cup S_r^{w^k} = \varnothing$, for all $k = 2, 3, ..., N$. This shows that $I_G\left(g_w^k\right) = 0$, for all $k = 2, 3, ..., N$. So, $I_G\left(g\right) = a_0\ I_G\left(1\right) + a_1\ I_G\left(g_w\right)$. ∎

**Corollary 3.6.** *Let $w = vwv$ be a loop finite path in $D_r(G^{\hat{}})$, with $v \in V(G^{\hat{}})$, and let $g = \sum_{n=0}^{N} a_n g_w^n$, with $g_w^0 = 1$, where $a_0, ..., a_n \in \mathbb{R}$. Then*

$$I_G\left(g\right) = a_0 \mu_{G^{\hat{}}}\left(D_r(G)\right) + \sum_{k=1}^{N} a_k \mu_{G^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right).$$

□

Now, we will consider the map $g_{-1}$ defined by $g_{-1}(x) = g_{x^{-1}} = g_{\delta^r_{x^{-1}}}$. On $D_r(G^{\hat{}})$, the map $g_{-1}$ is well-determined, because, for any $w \in \mathbb{F}^+(G^{\hat{}})$, there always exists $w^{-1}$. Also, the admissibility of $w^{-1}$ is preserved by that of $w$. This discussion says that the support $D_r(G^{\hat{}} : g_{\delta^r_w})$ of $g_{\delta^r_w}$ and the support $D_r(G^{\hat{}} : g_{\delta^r_{w^{-1}}})$ of $g_{\delta^r_{w^{-1}}}$ are same. i.e., If $w$ is in $D_r(G^{\hat{}})$, then

$$D_r(G^{\hat{}} : g_w) = D_r(G^{\hat{}} : g_{w^{-1}}), \text{ for all } w \in D_r(G^{\hat{}}). \tag{3.14}$$

Futhermore, we can get that

$$D_r(G^{\hat{}} : g_{-1}) = D_r(G^{\hat{}}) = D_r(G^{\hat{}} : g_1), \tag{3.15}$$

where $g_1(x) = g_x$, for all $x \in D_r(G^{\hat{}})$, since $g_{-1}$ has its support $D_r(G^{\hat{}})$.

**Proposition 3.7.** *Let $g_{-1}$ be given as above. Then $g_{-1}$ is $G^{\hat{}}$-measurable and $I_G\,(g_{-1}) = I_G\,(g_1)$.* $\square$

Similarly, we can conclude that;

**Proposition 3.8.** *Define $g_{-n}(x) = g_{\delta^r_{x^{-n}}}$, for $n \in \mathbb{N}$, for $x \in D_r(G^{\hat{}})$. Then $I_G\,(g_{-n}) = I_G\,(g_n)$.*

*Proof.* Observe that if $n \geq 2$, then

$$D_r(G^{\hat{}} : g_{-n}) = D_r(G^{\hat{}} : g_{-2}) = V(G^{\hat{}}) \cup loop_r(G^{\hat{}}) = D_r(G^{\hat{}} : g_2).$$

So, $I_G\,(g_{-n}) = I_G\,(g_{-2}) = I_G\,(g_2)$. By the previous proposition, $I_G\,(g_{-1}) = I_G\,(g_1)$. ∎

So, we can consider the trigonometric polynomials on $\mathbb{F}^+(G^{\hat{}})$.

**Theorem 3.9.** *Let $g_k$ and $g_{-k}$ be given as above, for all $k \in \mathbb{N}$, and let $g = \sum_{n=-M}^{N} a_n g_n$ be a trigonometric polynomial, with $g_0 \equiv 1$, where $N$, $M \in \mathbb{N}$. Then*

$$I_G\,(g) = a_0 \mu_{G^{\hat{}}}\,(D_r(G)) + \left(a_1 + a^{-1}\right) I_G(g_1) + \textstyle\sum_{n=-M}^{-2} a_n I_G\,(g_2) + \sum_{k=2}^{N} a_k I_G\,(g_2)\,. \tag{3.16}$$

*Proof.* Let $g$ be the given trigonometric polynomial on $D_r(G^{\hat{}})$. Then

$$I_G\,(g) = I_G\left(\textstyle\sum_{n=-N}^{N} a_n g_n\right) = \sum_{n=-N}^{N} a_n I_G\,(g_n)$$

$= \sum_{n=-M}^{-2} a_n I_G(g_n) + a_{-1} I_G(g_{-1}) + a_0 I_G(g_0) + a_1 I_G(g_1) + \sum_{k=2}^{N} a_k I_G(g_k)$

$= \sum_{n=-M}^{-2} a_n I_G(g_n) + a_0 \mu_{G^{\hat{}}}(D_r(G)) + + \sum_{k=1}^{N} a_k I_G(g_k)$

by the previous proposition

$= a_0 \mu_{G^{\hat{}}}(D_r(G)) + (a_{-1} + a_1) I_G(g_1) + \sum_{n=-M}^{-1} a_n I_G(g_2) + \sum_{k=1}^{N} a_k I_G(g_2)$

by the fact that $I_G(g_n) = I_G(g_2)$, for all $n \in \mathbb{N} \setminus \{1\}$. ∎

3.3. **Examples.**

In this section, we will consider certain finite directed graphs. Let $G_\Lambda$ be a tree with

$$V(G_\Lambda) = \{v_1, v_2, v_3\} \text{ and } E(G_\Lambda) = \{e_1 = v_1 e_1 v_2,\ e_2 = v_1 e_2 v_3\}.$$

Let $G_\Delta$ be a circulant graph with

$$V(G_\Delta) = \{v_1, v_2, v_3\} \text{ and } E(G_\Delta) = \left\{ \begin{array}{c} e_1 = v_1 e_1 v_2, \\ e_2 = v_2 e_2 v_3, \\ e_3 = v_3 e_3 v_1 \end{array} \right\}.$$

Also, assume that the corresponding reduced diagram graph measures are unweighted. i.e.,

$$W_\Lambda = 1 = W_\Delta,$$

where $W_\Lambda$ and $W_\Delta$ are the weighting map on $FP(G_\Lambda^{\hat{}})$ and $FP(G_\Delta^{\hat{}})$, respectively.

**Example 3.2.** *Consider $G_\Lambda$. We have that*

$$\begin{aligned} I_{G_\Lambda}(g_{v_1}) &= \mu_{G_\Lambda^{\hat{}}}\left(\{v_1, e_1, e_2, e_1^{-1}, e_2^{-1}\}\right) = d(\{v_1\}) + \Delta^{\hat{}}\left(\{e_1, e_2, e_1^{-1}, e_2^{-1}\}\right) \\ &= d(v_1) + \Delta^{\hat{}}(e_1) + \Delta^{\hat{}}(e_2) + \Delta^{\hat{}}\left(e_1^{-1}\right) + \Delta^{\hat{}}\left(e_2^{-1}\right) \\ &= \tfrac{4}{3} + 4 = \tfrac{16}{3}. \end{aligned}$$

$$\begin{aligned} I_{G_\Lambda}(g_{v_2}) &= \mu_{G_\Lambda^{\hat{}}}(\{v_2, e_1, e_1^{-1}\}) = d(\{v_2\}) + \Delta^{\hat{}}(\{e_1, e_1^{-1}\}) \\ &= \tfrac{2}{3} + 2 = \tfrac{8}{3}. \end{aligned}$$

*Similarly,* $I_{G_\Lambda}(g_{v_3}) = \frac{8}{3}$.

$$\begin{array}{ll} I_{G_\Lambda}\left(g_{e_1}\right) & = \mu_{G_\Lambda^{\hat{}}}\left(\{v_1, v_2, e_2^{-1}\}\right) = d\left(\{v_1, v_2\}\right) + \Delta^{\hat{}}\left(e_2^{-1}\right) \\ & = d(v_1) + d(v_2) + 1 = \frac{\deg(v_1)}{|V(G)|} + \frac{\deg(v_2)}{|V(G)|} + 1 \\ & = \frac{4}{3} + \frac{2}{3} + 1 = 3 = I_{G_\Lambda}\left(g_{e_1^{-1}}\right). \end{array}$$

*Similarly,* $I_{G_\Lambda}\left(g_{e_2}\right) = 3 = I_{G_\Lambda}\left(g_{e_2^{-1}}\right)$. *We have that*

$$D_r(G_\Lambda) = \{v_1, v_2, v_3\} \cup \{e_1, e_2, e_1^{-1}, e_2^{-1}\}.$$

*Thus*

$$\begin{array}{ll} I_{G_\Lambda}\left(g_1\right) & = \sum\limits_{w \in D_r(G_\Lambda^{\hat{}})} I_{G_\Lambda}\left(g_w\right) \\ & = \sum_{j=1}^{3} I_{G_\Lambda}\left(g_{v_j}\right) + \sum_{k=1}^{2} I_{G_\Lambda}\left(g_{e_k}\right) + \sum_{i=1}^{2} I_{G_\Lambda}\left(g_{e_i^{-1}}\right) \\ & = \left(\frac{16}{3} + \frac{8}{3} + \frac{8}{3}\right) + (2+2) + (2+2) = \frac{56}{3}. \end{array}$$

*We also have that*

$$I_{G_\Lambda}\left(g_n\right) = \sum_{j=1}^{3} I_{G_\Lambda}\left(g_{v_j}\right) = \frac{16}{3} + \frac{8}{3} + \frac{8}{3} = \frac{32}{3},$$

*for all* $n \in \mathbb{N} \setminus \{1\}$, *because* $w^k = \emptyset$, *for all* $w \in D_r^{FP}(G_\Lambda^{\hat{}})$ *and* $k \in \mathbb{N} \setminus \{1\}$, *and* $D_r(G_\Lambda^{\hat{}} : g_k) = V(G_\Lambda^{\hat{}}) \cup loop_r(G_\Lambda^{\hat{}}) = V(G_\Lambda^{\hat{}})$. *Therefore,*

$$I_{G_\Lambda}\left(g_p\right) = \frac{56}{3} + \frac{32(N-1)}{3}.$$

*where* $g_p = \sum_{n=1}^{N} g_n$, *with* $g_0 = 0$, *for all* $N \in \mathbb{N}$.

**Example 3.3.** *Consider the graph* $G_\Delta$. *Then we have;*

$$\begin{array}{ll} I_{G_\Delta}\left(g_{v_1}\right) & = \mu_{G_\Delta^{\hat{}}}\left(\{v_1, e_1^{\pm 1}, e_3^{\pm 1}\}\right) = d(\{v_1\}) + \Delta^{\hat{}}\left(\{e_1^{\pm 1}, e_3^{\pm 1}\}\right) \\ & = \frac{\deg(v)}{|V(G)|} + 2L\left(e_1\right) + 2L(e_3) = \frac{4}{3} + 2 + 2 \\ & = \frac{16}{3}. \end{array}$$

*Similarly,* $I_{G_\Delta}\left(g_{v_2}\right) = \frac{16}{3} = I_{G_\Delta}\left(g_{v_3}\right)$. *By the existence of the circulant parts, we can have that*

$$I_{G_\Delta}\left(g_{e_j}\right) = \frac{17}{3} = I_{G_\Delta}\left(g_{e_j^{-1}}\right), \text{ for all } j = 1,\ 2,\ 3.$$

*Let* $j = 1$. *Then* $I_{G_\Delta}\left(g_{e_1}\right) = \mu_{G_\Delta^{\hat{}}}\left(\delta^r(S_l^w) \cup \delta^r(S_r^w)\right)$, *where*

$$S_l^w = \{v_2, e_2\} \quad \text{and} \quad S_r^w = \{v_1, e_1^{-1}, e_3\},$$

*so,*

$$S_l^w \cup S_r^w = \{v_1, v_2, e_1^{-1}, e_2, e_3\}.$$

*Thus*

$$I_{G_\Delta}(g_{e_1}) = \mu_{G_\Delta^{\hat{}}}\left(\{v_1, v_2, e_1^{-1}, e_2, e_3\}\right)$$

$$= d(\{v_1, v_2\}) + \Delta^{\hat{}}\left(\{e_1^{-1}, e_2, e_3\}\right)$$

$$= \frac{\deg(v_1)}{|V(G)|} + \frac{\deg(v_2)}{|V(G)|} + L\left(e_1^{-1}\right) + L(e_2) + L(e_3)$$

$$= \frac{4}{3} + \frac{4}{3} + 1 + 1 + 1 = \frac{17}{3}.$$

*Now, consider the element* $l = e_1 e_2 e_3$ *in* $D_r(G^{\hat{}})$. *Then*

$$\delta^r(S_l^l) = \{v_1, l\} \quad \text{and} \quad \delta^r(S_r^l) = \{v_1, l\},$$

*so,*

$$\delta^r(S_l^l) \cup \delta^r(S_r^l) = \{v_1, l\}.$$

*Therefore,*

$$I_{G_\Delta}(g_l) = \mu_{G^{\hat{}}}(\{v_1, l\}) = \frac{\deg(v_1)}{|V(G)|} + L(l)$$

$$= \frac{4}{3} + 3 = \frac{13}{3}.$$

## 4. Subgraph Measure Theory

In this chapter, we will consider a new measure on the finite directed graph. Throughout this chapter, let $G$ be a finite directed graph and $G^{\hat{}}$, the corresponding shadowed graph and let $\mathbb{F}^+(G^{\hat{}})$ be the free semigroupoid of $G^{\hat{}}$. Suppose $H$ is a full subgraph of $G$. The shadowed graph $H^{\hat{}}$ of $H$ is also a subgraph of the shadowed graph $G^{\hat{}}$ of $G$. So, canonically, we can construct the $H$-measures $\mu_H$, as in Chapter 1, and the $H^{\hat{}}$-measure $\mu_{H^{\hat{}}}$, as in Chapter 2. We restrict our interests to the $H^{\hat{}}$-measure $\mu_{H^{\hat{}}}$.

**Definition 4.1.** *Let* $(G^{\hat{}}, \mu_{G^{\hat{}}})$ *be the reduced diagram graph measure space of* $G$ *and let* $H$ *be a full subgraph of* $G$. *Define the* $H^{\hat{}}$*-measure* $\mu_{H^{\hat{}}}$ *by* $d_H \cup \Delta_H^{\hat{}}$, *where*

$$d_H(S) \overset{def}{=} \sum_{v \in V(H) \cap S_V} \frac{\deg(v)}{|V(H)|}, \tag{4.1}$$

*for all* $S_V \subseteq V(G^{\hat{}})$, *and*

$$\Delta_H^{\hat{}}(S_{FP}) \overset{def}{=} \sum_{w \in FP(H) \cap S_{FP}} L(w) = \sum_{w \in FP(H) \cap S_{FP}} W(\delta_w^r) f(\delta_w^r), \tag{4.2}$$

*for all* $S_{FP} \subseteq D_r^{FP}(G)$.

By definition, the subgraph measure $\mu_{H^{\hat{}}}$ is the restricted measure $\mu_{G^{\hat{}}}\ |_{D_r(H^{\hat{}})}$ .

**Proposition 4.1.** *Let* $g_w = 1_{\delta^r(S_l^w)\cup\delta^r(S_r^w)}$ *be the* $G^{\hat{}}$*-measurable function defined in Section 3.2, for each* $w \in D_r(G^{\hat{}})$. *Then graph integral* $I_{G:H}\,(g_w)$ *of* $g_w$ *with respect to the subgraph measure* $\mu_{H^{\hat{}}}$ *is determined by*

$$\begin{aligned} I_{G:H}\,(g_w) &= \mu_{H^{\hat{}}}\,(\delta^r(S_l^w) \cup \delta^r(S_r^w)) \\ &= \mu_{G^{\hat{}}}\left(D_r(H^{\hat{}}) \cap (\delta^r(S_l^w) \cup \delta^r(S_r^w))\right). \end{aligned}$$

□

## 5. Extended Graph Measures

In this chapter, based on the graph measure construction in Chapter 2, we will consider the extended graph measuring. Throughout this chapter, let $G$ be a finite directed graph and $G^{\hat{}}$, the shadowed graph and let $\mathbb{F}^+(G^{\hat{}})$ and $D_r(G^{\hat{}})$ be the corresponding free semigroupoid of $G^{\hat{}}$ and the reduced diagram set of $G^{\hat{}}$, respectively. As usual, let $\delta^r : \mathbb{F}^+(G^{\hat{}}) \to D_r(G^{\hat{}})$ be the reduced diagram map which is a surjection. By definition, the power set $P\left(\mathbb{F}^+(G^{\hat{}})\right)$ is a $\sigma$-algebra of $\mathbb{F}^+(G^{\hat{}})$. Indeed,

(i) $\varnothing, \mathbb{F}^+(G^{\hat{}}) \in P\left(\mathbb{F}^+(G^{\hat{}})\right)$
(ii) If $S \subseteq \mathbb{F}^+(G^{\hat{}})$, then $\mathbb{F}^+(G^{\hat{}}) \setminus S \subseteq \mathbb{F}^+(G^{\hat{}})$
(iii) If $(S_k)_{k=1}^{\infty} \subseteq P\left(\mathbb{F}^+(G^{\hat{}})\right)$, then $\cup_{k=1}^{\infty} S_k \subseteq \mathbb{F}^+(G^{\hat{}})$.

We want to define a suitable measure on $\mathbb{F}^+(G^{\hat{}})$.

**Definition 5.1.** *Define a measure* $\mu$ *on* $\mathbb{F}^+(G^{\hat{}})$ *by* $d \cup \omega$, *where* $d$ *is the weighted degree measure on* $V(G^{\hat{}})$ *defined in Chapter 2. i.e.,*

$$d\,(S_V) = \sum_{v \in S_V} \frac{\deg(v)}{|V(G^{\hat{}})|},$$

*and* $\omega$ *is the measure on* $FP(G^{\hat{}})$ *defined by*

$$\omega(S_{FP}) = \sum_{w \in S_{FP}} W(\delta_w^r) f(\delta_w^r),$$

*where $W$ and $f$ are the weighting map and the reduced diagram length map defined on $D_r(G^{\hat{}})$, in Chapter 2. The measure $\mu$ is called the (extended) graph measure on $\mathbb{F}^+(G^{\hat{}})$.*

It is easy to check that $\mu$ is a measure on $\mathbb{F}^+(G^{\hat{}})$. Remark the difference between $\mu_{G^{\hat{}}}$ on $D_r(G^{\hat{}})$ and $\mu$ on $\mathbb{F}^+(G^{\hat{}})$. A difference between $\mu$ and $\mu_{G^{\hat{}}}$ is recognized by the following example. Suppose $l$ is a loop-edge in $E(G^{\hat{}})$ and let $S_1 = \{l\}$ and $S_2 = \{l^2\}$. Then the union $S = S_1 \cup S_2$ is in $\mathbb{F}^+(G^{\hat{}})$, but $S$ is not in $D_r(G^{\hat{}})$. So, the measure $\mu_{G^{\hat{}}}(S)$ is undefined. However, it is defined on $\delta^r(S) = \{\delta_l^r,\, \delta_{l^2}^r\} = \{l\}$. Moreover, we have that

$$\begin{aligned}
\mu_{G^{\hat{}}}\,(\delta^r(S)) &= \mu_{G^{\hat{}}}\,(\delta^r(S_1) \cup \delta^r(S_2)) \\
&= \Delta^{\hat{}}\,(\delta^r(S_1) \cup \delta^r(S_2)) \\
&= \Delta^{\hat{}}\,(\{l\} \cup \{l\}) = \Delta^{\hat{}}\,(\{l\}) = W(l)
\end{aligned}$$

and

$$\begin{aligned}
\mu\,(S) &= \mu(S_1 \cup S_2) = \mu(\{l, l^2\}) = \omega\left(\{l, l^2\}\right) \\
&= \omega(\{l\}) + \omega(\{l = \delta_{l^2}^r\}) = 2W(l).
\end{aligned}$$

This difference shows us that the measure $\mu$ is not bounded, in general. However, this measure $\mu$ is locally bounded in the sense that $\mu(S) < \infty$, for all finite subsets $S$ in $\mathbb{F}^+(G^{\hat{}})$.

**Proposition 5.1.** *The extended graph measure $\mu$ on $\mathbb{F}^+(G^{\hat{}})$ is a locally bounded positive measure.* □

**Definition 5.2.** *The triple $\left(\mathbb{F}^+(G^{\hat{}}),\ P\left(\mathbb{F}^+(G^{\hat{}})\right),\ \mu\right)$ is called the extended graph measure space (in short, $\mu$-measure space). For convenience, we denote it by $(G^{\hat{}}, \mu)$.*

In the following section, we will show that the similar integration theory holds for the extended graph measure like in Chapter 3.

### 5.1. **Integration on Graphs.**

In this section, we will define the integrals of the given $\mu$-measurable functions, with respect to the extended graph measure $\mu$. First, let

$$g = \textstyle\sum_{n=1}^{N} a_n 1_{S_n}, \text{ for all } S_n \subseteq \mathbb{F}^+(G^{\hat{}}), \tag{5.1}$$

where $a_1$, ..., $a_N \in \mathbb{R}$, be a simple function. Then the integral $I_\mu(g)$ of $g$ with respect to $\mu$ is defined by

$$I_G(g) \overset{denote}{=} \int_{G^{\hat{}}} g \; d\mu \overset{def}{=} \sum_{n=1}^{N} a_n \; \mu(S_n). \tag{5.2}$$

Since $\mu = d \cup \omega$, the definition (5.2) can be rewritten as

$$\begin{aligned} I_G(g) &= \sum_{n=1}^{N} a_n \mu_{G^{\hat{}}}(S_n) = \sum_{n=1}^{N} a_n \left( d(S_{n,V}) + \omega(S_{n,FP}) \right) \\ &= \sum_{n=1}^{N} a_n \left( \sum_{v \in S_{n,V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{n,FP}} W(\delta_w^r) f(\delta_w^r) \right). \end{aligned}$$

On the set of all simple functions, we can determine the "almost everywhere" relation (in short, "a.e" relation).

**Proposition 5.2.** *Let $g_1 = \sum_{j=1}^{n} 1_{S_j}$ and $g_2 = \sum_{i=1}^{m} 1_{T_i}$ be simple functions, where $S_j$'s and $T_i$'s are subsets of $\mathbb{F}^+(G^{\hat{}})$. Suppose that $S_j$'s are mutually disjoint and also $T_i$'s are mutually disjoint. If $\cup_{j=1}^{n} S_j = \cup_{i=1}^{m} T_i$ in $\mathbb{F}^+(G^{\hat{}})$, then $I(g_1) = I(g_2)$.*

*Proof.* Observe that

$$I(g_1) = \sum_{j=1}^{n} \left( \sum_{v \in S_{j,V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{j,FP}} W(\delta_w^r) f(\delta_w^r) \right)$$

and

$$I(g_2) = \sum_{i=1}^{m} \left( \sum_{v \in T_{i,V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in T_{i,FP}} W(\delta_w^r) f(\delta_w^r) \right).$$

Since $\{S_j\}_{j=1}^{n}$ and $\{T_i\}_{i=1}^{m}$ are disjoint families of $P\left(\mathbb{F}^+(G^{\hat{}})\right)$,

$$\sum_{j=1}^{n} \sum_{v \in S_{j,V}} \frac{\deg(v)}{|V(G)|} = \sum_{v \in \cup_{j=1}^{n} S_{j,V}} \frac{\deg(v)}{|V(G)|}$$

and

$$\sum_{i=1}^{m} \sum_{v \in T_{i,V}} \frac{\deg(v)}{|V(G)|} = \sum_{v \in \cup_{i=1}^{m} T_{i,V}} \frac{\deg(v)}{|V(G)|},$$

by the mutually disjointness on $\{S_{j,V}\}_{j=1}^{n}$ and $\{T_{i,V}\}_{i=1}^{m}$. Similarly, by the mutually disjointness of $\{S_{j,FP}\}_{j=1}^{n}$ and $\{T_{i,FP}\}_{i=1}^{m}$ and by (3.7),

$$\sum_{j=1}^{n} \sum_{w \in S_{j,FP}} \omega(\delta_w^r) = \sum_{w \in \cup_{j=1}^{n} S_{j,FP}} \omega(\delta_w^r)$$

and

$$\sum_{i=1}^{m} \sum_{w \in T_{i,FP}} \omega(\delta_w^r) = \sum_{w \in \cup_{i=1}^{m} T_{i,FP}} \omega(\delta_w^r).$$

Therefore, we can conclude that $I(g_1) = I(g_2)$, by (3.5). ∎

It is easy to check that, if $g_j = \sum_{k=1}^{n} a_{j,k} 1_{S_{j,k}}$, for $j = 1, 2$, then we can have

$$I_G(g_1 + g_2) = I_G(g_1) + I_G(g_2) \tag{5.3}$$

$$I_G(c \cdot g_j) = c \cdot I_G(g_j), \text{ for all } j = 1, 2. \tag{5.4}$$

Now, let $1_S$ and $1_T$ be the characteristic functions, where $S \neq T$ in $P\left(\mathbb{F}^+(\hat{G})\right)$. Then we have that $1_S \cdot 1_T = 1_{S \cap T}$.

**Proposition 5.3.** *Let $g_j$ be given as above, for $j = 1, 2$. Then*

$$I_G(g_1 g_2) = \sum_{k,i=1}^{n} (a_{1,k} a_{2,i}) \cdot \left( \sum_{v \in S_{(1,k),V} \cap S_{(2,i),V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{(1,k),FP} \cap S_{(2,i),FP}} \omega(\delta_w^r) \right).$$

*Proof.* Observe that

$$g_1 g_2 = \left(\sum_{k=1}^{n} a_{1,k} 1_{S_{1,k}}\right) \left(\sum_{i=1}^{n} a_{2,i} 1_{S_{2,i}}\right)$$

$$= \sum_{k,i=1}^{n} a_{1,k} a_{2,i} \left(1_{S_{1,k}} \cdot 1_{S_{2,i}}\right) = \sum_{k,i=1}^{n} a_{1,k} a_{2,i} 1_{S_{1,k} \cap S_{2,i}}$$

So, we have that

$$I(g_1 g_2) = \sum_{k,i=1}^{n} a_{1,k} a_{2,i} \quad \mu(S_{1,k} \cap S_{2,i}). \tag{5.5}$$

Consider

$$\mu(S_{1,k} \cap S_{2,i}) = d\left((S_{1,k} \cap S_{2,i})_V\right) + \omega\left((S_{1,k} \cap S_{2,i})_{FP}\right)$$

$$= d\left(S_{(1,k),V} \cap S_{(2,i),V}\right) + \omega\left(S_{(1,k),FP} \cap S_{(2,k),FP}\right)$$

$$= \sum_{v \in S_{(1,k),V} \cap S_{(2,i),V}} \frac{\deg(v)}{|V(G)|} + \sum_{w \in S_{(1,k),FP} \cap S_{(2,i),FP}} \omega(\delta_w^r).$$

∎

Suppose that $g_1$ and $g_2$ are given as above and assume that the families $\{S_{1,k}\}_{k=1}^{n}$ and $\{S_{2,i}\}_{i=1}^{n}$ are disjoint family. i.e.,

$$(\cup_{k=1}^{n} S_{1,k}) \cap (\cup_{i=1}^{n} S_{2,i}) = \varnothing.$$

Then $I(g_1 g_2) = 0$.

Similar to Chapter 3, we can define a $\mu$-measurable function $g_w$, for all $w \in \mathbb{F}^+(G^{\hat{}})$, by

$$g_w = 1_{S_l^w \cup S_r^w},$$

where

$$S_l^w = \{w' \in \mathbb{F}^+(G^{\hat{}}) : ww' \in \mathbb{F}^+(G^{\hat{}})\}$$

and

$$S_r^w = \{w'' \in \mathbb{F}^+(G^{\hat{}}) : w''w \in \mathbb{F}^+(G^{\hat{}})\}.$$

**Proposition 5.4.** *Let $w \in \mathbb{F}^+(G^{\hat{}})$ and assume that $g_w$ is the above $\mu$-measurable map. Then*

$$I(g_w) = \mu\left(S_l^w \cup S_r^w\right), \tag{5.6}$$

*where $S_l^w$ and $S_r^w$ are given in the previous paragraph.* $\square$

So, different from Chapter 3, if $w$ is a loop finite path in $\mathbb{F}^+(G^{\hat{}})$, then the integral $I(g_w)$ of $w$ is $\infty$, whenever the weight $W(\delta_w^r)$ is large enough in the interval (0, 1), because both $S_l^w$ and $S_r^w$ contains $w^k$, for all $k \in \mathbb{N}$.

Let $g$ be a $\mu$-measurable function. Then the support $\mathbb{F}^+(G^{\hat{}} : g)$ of $g$ is well-defined. Define the monomial $g_k(w) = g_{w^k}$ on $\mathbb{F}^+(G^{\hat{}})$, for $k \in \mathbb{N}$. Then

$$\mathbb{F}^+(G^{\hat{}} : g_1) = \mathbb{F}^+(G^{\hat{}})$$

and

$$\mathbb{F}^+(G^{\hat{}} : g_n) = V(G^{\hat{}}) \cup loop(G^{\hat{}}),$$

for all $n \in \mathbb{N} \setminus \{1\}$, where

$$loop(G^{\hat{}}) \overset{def}{=} \{l \in \mathbb{F}^+(G^{\hat{}}) : l \text{ is a loop finite path}\}.$$

So, by (5.6) and the supports, $I(g_k) = \infty = I(g)$, for $k \in \mathbb{N}$, in general, where $g = \sum_{k=0}^{N} a_k\, g_k$ is a polynomial, with $g_0 \equiv 1$.

Define the $\mu$-measurable map $g_{-n}(w) = g_{w^{-n}}$, for all $w \in \mathbb{F}^+(G^{\hat{}})$ and $n \in \mathbb{N}$. Then similar to Chapter 3, we have that

$$\mathbb{F}^+(G^{\hat{}} : g_{-n}) = \mathbb{F}^+(G^{\hat{}} : g_n), \text{ for } n \in \mathbb{N}. \tag{5.7}$$

$$\mathbb{F}^+(G^{\hat{}} : g_{\pm n}) = \mathbb{F}^+(G^{\hat{}} : g_2), \text{ for all } n \in \mathbb{N} \setminus \{1\}. \tag{5.8}$$

By (5.7) and (5.8), we have the following proposition.

**Proposition 5.5.** *$I(g_{-n}) = I(g_n)$, for all $n \in \mathbb{N}$. In particular, $I(g_{\pm n}) = I(g_2)$, for all $n \in \mathbb{N} \setminus \{1\}$.* $\square$

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